Advanced Mathematical Models & Applications
Vol.4, No.3, 2019, pp.210-223

MARICHEV-SAIGO-MAEDA FRACTIONAL CALCULUS OPERATORS WITH EXTENDED MITTAG-LEFFLER FUNCTION AND GENERALIZED GALUE TYPE STRUVE FUNCTION

Rameez Aziz[1*], **Yaghvendra Kumawat**[2]

[1,2]Department of Mathematics, Vivekananda Global University, Jaipur, India

Abstract. In this paper several fractional calculus operators have been introduced and investigated. The aim is to establish the Marichev-Saigo-Maeda (MSM) fractional calculus operators and Caputo-type MSM fractional differential operators involving the product of extended Mittag-Leffler function (EMLF) and generalized Galue Struve Type Function. Some of the particular cases of the main results are also derived. The results given in this paper are general in character and likely to find some applications in the theory of special functions.

Keywords: Marichev-Saigo-Maeda fractional integral operators, generalized Mittag-Leffler function, generalized Galue Stuve Type Function, generalized hypergeometric series, fractional derivative operators.

AMS Subject Classification: 26A33, 33E12, 33C20, 33C60.

Corresponding author: Rameez Aziz, Department of Mathematics, Vivekananda Global University, Jaipur, India, Tel.: +91-9796716240, e-mail: *sraziz11@gmail.com*

Received: 22 July 2019; Revised: 07 October 2019; Accepted: 22 November 2019;
Published: 21 December 2019.

1 Introduction

Fractional calculus is a very fast developing subject of mathematics which deals with the study of fractional order derivatives and integrals. Many applications of fractional calculus can be found in image processing, nonlinear biological systems, fluid dynamics, stochastic dynamical systems, nonlinear control theory, plasma physics and controlled thermonuclear fusion and in quantum mechanics. Fractional calculus is a proficient tool to study many complex real problems (Hilfer, 2000). The fractional integral operator has many interesting applications in various subfields in applicable mathematical analysis. The results given in (Miller & Ross, 1993; Kiryakova, 1997; Srivastava et al., 2006) can be referred to for some basic results on fractional calculus. During the past four decades, a number of researches have studied the properties, applications, and different extensions of various operators of fractional calculus (Marichev, 1974; Oldham & Spanier, 1974; Kiryakova, 1993, 2006; Kilbas et al., 2006). A useful generalized of hypergeometric fractional integrals, including the Saigo operators (Saigo, 1978, 1979, 1980) has been introduced by (Marichev, 1974) and later extended and studied by Saigo and Maeda [(Saigo & Maeda, 1998), p.393, eqn. (4.12) and (4.13)] in terms of any complex order with Appell's function F3(.) in the kernel as follows:

Let $\eta, \eta', \sigma, \sigma', \varrho \in \mathbb{C}$ *and* $x > o$; then the generalized fractional calculus operators (MSM operators) involving the Appel Function are defined by the following equations

$$\left(I_{0,+}^{\eta,\eta',\sigma,\sigma',\varrho} f\right)(x) = \frac{x^{-\eta}}{\Gamma(\varrho)} \int_0^x (x-t)^{\varrho-1} t^{-\eta'} F_3\left(\eta, \eta', \sigma, \sigma'; \varrho; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt \qquad (1)$$

and

$$\left(I_{0,-}^{\eta,\eta',\sigma,\sigma',\varrho}f\right)(x)=\frac{x^{-\eta'}}{\Gamma(\varrho)}\int_0^{\infty}(t-x)^{\varrho-1}t^{-\eta}F_3\left(\eta,\eta',\sigma,\sigma';\varrho;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt \tag{2}$$

with $\mathbb{R}(\varrho)>0$.

The generalized fractional integral operators of (1) and (2) are introduced by (Marichev, 1974) and later was studied and extended by Saigo and Maeda (Saigo & Maeda, 1998) and this fractional integral operator is known as Marichev-Saigo-Maeda Operators (MSM operator). In (1) and (2), F_3 denotes the 3rd Appell function (also known as Horn Function) (Srivastava & Karlson, 1985)

$${}_pF_q(\eta,\eta',\sigma,\sigma';\gamma;x;y)=\sum_{n=0}^{\infty}\frac{(\eta)_m(\eta')_n(\sigma)_m(\sigma')_n}{(\gamma)_{m+n}\ m!\ n!}x^my^n;\ \max\{|x|,|y|\}<1.$$

In recent times, many researchers have studied the image formulas for Marichev-Saigo-Maeda (MSM) fractional integral operators relating different special functions.

The resultant fractional differential operators have their particular forms:

$$\left(D_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma}f\right)(x)=\left(\frac{d}{dx}\right)^{[\Re(\gamma)]+1}\left(I_{0,+}^{-\eta',-\eta,-\sigma'+[\Re(\gamma)]+1,-\sigma,-\gamma+[\Re(\gamma)]+1}f\right)(x) \tag{3}$$

and

$$\left(D_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma}f\right)(x)=\left(-\frac{d}{dx}\right)^{[\Re(\gamma)]+1}\left(I_{0,-}^{-\eta',-\eta,-\sigma',-\sigma+[\Re(\gamma)]+1,-\gamma+[\Re(\gamma)]+1}f\right)(x). \tag{4}$$

The Mittag Leffler function was introduced in (Mittag-Leffler, 1903) as

$$E_\rho(x)=\sum_{n=o}^{\infty}\frac{x^n}{\Gamma(\rho n+1)}\qquad (x\in\mathbb{C};\Re(\rho)>0) \tag{5}$$

(Wiman, 1905) defined as following the generalized form of Mittag Leffler function and applied to various fields (Dorrego et al., 2012; Gorenflo et al., 1998; Rahman et al., 2017)

$$E_{\rho,\sigma}(x)=\sum_{n=o}^{\infty}\frac{x^n}{\Gamma(\rho n+\sigma)}\qquad (x,\sigma\in\mathbb{C};\Re(\rho)>0). \tag{6}$$

(Prabhakar, 1971) defined the generalized M-L function as

$$E_{\rho,\sigma}^{\gamma}(x)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\rho n+\sigma)}\frac{x^n}{n!}\qquad (x,\sigma,\gamma\in\mathbb{C};\Re(\rho)>0), \tag{7}$$

where $(\gamma)_n$ denotes the Pochhammer symbol defined $(\gamma\in\mathbb{C})$, in the terms of the familiar gamma function Γ [(Srivastava & Choi, 2012), p.2 and p.5] by

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1 & (n=0;\ \gamma\subset\{0\}\\ \gamma(\gamma+1)\cdot(\gamma+n-1) & (n\in N;\ \gamma\in\mathbb{C}\end{cases}.$$

(Ozarslan & Yilmaz, 2014) investigated and introduced the extended Mittag-Leffler function as following

$$E_{\delta,\nu}^{\lambda,\mu}(t;p)=\sum_{n=0}^{\infty}\frac{B_p(\lambda+n,\mu-\lambda)}{B(\lambda,\mu-\lambda)}\frac{(\mu)_n}{\Gamma(\delta_n+\nu)}\frac{t^n}{n!}\qquad (t,\nu\in\mathbb{C};p\geqslant 0;\Re(\nu)>\Re(\lambda)>0,\Re(\delta)>0), \tag{8}$$

where $B_p(m,n)$ is the extended beta function (Chaudhry et al., 1997) defined by

$$B_p(m,n) = \int_0^1 U^{m-1}(1-u)^{n-1} e^{-\frac{p}{u(1-u)}} du \qquad (min\{\Re(p), \Re(m), \Re(y)\} > 0).$$

$B_0(m,n) = B(m,n)$ is the familiar beta function given by (see, section 1.1 (Srivastava & Choi, 2012))

$$B(m,n) = \begin{cases} \int\limits_0^1 u^{m-1}(1-u)^{n-1} du & (\min\{\Re(m), \Re(n)\} > 0), \\ \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} & (m,n \in \mathbb{C}/Z_0^-). \end{cases}$$

The generalized hypergeometric series ${}_pF_q$ is defined by (see, section 1.5 (Srivastava & Choi, 2012))

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix} x\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{x^n}{n!} \\ &=_p F_q(\alpha_1,\ldots,\alpha_p; \beta_1,\ldots,\beta_q; x). \end{aligned}$$

(Sharma & Devi, 2015) introduced and studied the extended Wright generalized hypergeometric function as

$$ {}_{m+1}\Psi_{n+1}\left[\begin{matrix}(a_i, A_i)_{1,m}, & (\gamma,1); \\ (b_j, B_j)_{1,n}, & (c,1);\end{matrix} x; p\right] = \frac{1}{\Gamma(c-\gamma)} \sum_{k=0}^{\infty} \frac{\Gamma(a_i + kA_i)}{\Gamma(b_j + kB_j)} \frac{B_p(\gamma+k, c-\gamma)x^k}{k!} \tag{9}$$

$(\Re(p) > 0, \Re(c) > \Re(\gamma) > 0; m,n \in N_0; a_i, b_j \in \mathbb{C}, A_I, B_j \in \Re^+; i = 1,\ldots,m; j = 1,\ldots,n)$ *with*

$$\sum_{j=1}^{n} B_j - \sum_{i=1}^{n} A_j > -1.$$

(Galue, 2003) introduced a generalization of the Bessel function of order h given by

$$ {}_\delta J_h(\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\delta k + h + 1)k!} \left(\frac{\xi}{2}\right)^{2k+h} \quad ; \xi \in \Re; \delta \in N = \{1,2,\ldots\}. \tag{10}$$

(Baricz, 2010) investigated Galue-type generalization of modified Bessel function as

$$ {}_\delta J_h(\xi) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\delta k + h + 1)k!} \left(\frac{\xi}{2}\right)^{2k+h} \quad ; \xi \in \Re; \delta \in N. \tag{11}$$

The Struve function of order h is given by

$$\mathcal{H}_h(\xi) = \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+h+\frac{3}{2}\right)} \left(\frac{\xi}{2}\right)^{2k+h} \quad ; \xi \in \Re; \delta \in N \tag{12}$$

is a particular solution of the non-homogeneous Bessel differential equation

$$\xi^2\, y''(\xi) + \xi\, y'(\xi) + (\xi^2 - h^2)\, y(\xi) = \frac{4\left(\frac{\xi}{2}\right)^{h+1}}{\sqrt{\pi}\Gamma\left(h+\frac{1}{2}\right)},$$

where Γ is the classical gamma function whose Euler's integral is given by Srivastava and Choi (Samko et al., 1993)

$$\Gamma(\xi)=\int_0^\infty e^{-t}\, t^{\xi-1}dt \quad ;\Re(\xi)>0.$$

The generalizations of Struve function are found in (Bhownick, 1962, 1963; Kanth, 1981; Nisar et al., 2016; Singh, 1974, 1985, 1988, 1989).
Another generalization of Struve function given by (Orhan & Yagmur, 2013, 2014) is

$$\mathcal{H}_{h,b,c}(\xi)=\sum_{k=0}^{\infty}\frac{(-c)^k}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+h+\frac{b}{2}+1\right)}\left(\frac{\xi}{2}\right)^{2k+h+1} \quad ;h,b,c\in\mathbb{C}. \tag{13}$$

The generalized Galue Type Struve function (GTSF) was recently defined by (Nisar et al., 2016) as

$${}_a\mathcal{W}^{\alpha,\beta}_{p,b,c,\xi}(z)=\sum_{k=0}^{\infty}\frac{(-c)^k}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{p}{\xi}+\frac{b+2}{2}\right)}\left(\frac{z}{2}\right)^{2k+p+1} \quad ;a\in N;p,b,c\in\mathbb{C}, \tag{14}$$

where $\alpha>0,\ \xi>0$ and β is arbitrary parameter.
When $\alpha=a=1,\ \beta=\frac{3}{2}$ *and* $\xi=1$ in above equation, it turns to the generalization of Struve function defined by (Orhan & Yagmur, 2013, 2014)

$$\mathcal{H}_{p,b,c}(z)=\sum_{k=0}^{\infty}\frac{(-c)^k}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+p+\frac{b+c}{2}\right)}\left(\frac{z}{2}\right)^{2k+p+1} \quad ;p,b,c\in\mathbb{C}.$$

Details related to the function $\mathcal{H}_{h,b,c}(z)$ and its particular cases can be seen in (Baricz, 2010; Mondal & Swaminathan, 2012; Mondal & Nisar, 2014; Nisar et al., 2016).

2 Marichev-Saigo-Maeda fractional integral representation involving product of Extended Mittag-Leffler Function and generalized Galue Type Struve Function

Here we present product of Extended Mittag-Leffler Function (EMLF) and generalized Galue Type Struve Function (GTSF) in view of the MSM fractional integral representations and consider some particular cases.

We recall the following lemmas [see (Saigo & Maeda, 1998) and (Kataria et al., 2015)]

Lemma 1. *Let* $\eta,\eta',\sigma,\sigma',\gamma,\rho\in\mathbb{C}$ *such that* $\Re(\gamma)>0$ *and* $\Re(\rho)>max\{0,\Re(\eta-\eta'-\sigma-\gamma),\Re(\eta'-\sigma'\}$. *Then*

$$\left(I^{\eta,\eta',\sigma,\sigma',\gamma}_{0,+}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\eta-\eta'-\sigma)\Gamma(\rho+\sigma'-\eta')}{\Gamma(\rho+\sigma')\Gamma(\rho+\gamma-\eta-\eta')\Gamma(\rho+\gamma-\eta'-\sigma)}x^{\rho-\eta-\eta'+\gamma-1}. \tag{15}$$

Lemma 2. *Let* $\eta,\eta',\sigma,\sigma',\gamma,\rho\in\mathbb{C}$ *such that* $\Re(\gamma)>0$ *and* $\Re(\rho)>max\{\Re(\sigma),\Re(-\eta-\eta'+\gamma),\Re(-\eta-\sigma'+\gamma\}$. *Then*

$$\left(I^{\eta,\eta',\sigma,\sigma',\gamma}_{0,-}t^{-\rho}\right)(x)=\frac{\Gamma(\rho-\sigma)\Gamma(\eta+\eta'-\gamma+\sigma)\Gamma(\eta+\sigma'-\gamma+\rho)}{\Gamma(\rho)\Gamma(\eta-\sigma+\rho)\Gamma(\eta+\eta'+\sigma'-\gamma+\rho)}x^{-\eta-\eta'+\gamma-\rho}. \tag{16}$$

Theorem 1. *Let* $\eta,\eta',\sigma,\sigma',q,r,s,c,\vartheta,\gamma,\rho\in\mathbb{C}$ *with* $\Re(\gamma)>0$ *and* $\Re(\rho)>max\{0,\Re(\eta+\eta'+\sigma-\gamma),\Re(\eta-\sigma')\}$ *and* $p\geqslant 0,\alpha>0,\xi>0;a$ *and* β *is arbitrary and let* $x\in\Re^+$. *Then*

$$\left(I^{\eta,\eta',\sigma,\sigma',\gamma}_{0,+}t^{\rho-1}E^{\lambda,c}_{\theta,\vartheta}(t,p)_a\mathcal{W}^{\alpha,\beta}_{q,r,s,\xi}(t)\right)(x)=\frac{(x)^{\rho+q-\eta-\eta'+\gamma}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$

$$\times{}_5\Psi_5\left[\begin{matrix}(c,1),(\rho+2k+q+1,1),(\rho+2k+q+1+\gamma-\eta-\eta'-\sigma,1),(\rho+2k+q+1+\sigma'-\eta',1),(\lambda,1);\\((\vartheta,\theta),(\rho+2k+q+1+\sigma',1),(\rho+2k+q+1+\gamma-\eta-\eta',1),(\rho+2k+q+1+\gamma-\eta'-\sigma,1),(c,1);\end{matrix}x;p\right] \tag{17}$$

Proof. Let the left-hand side of (17) be denoted by $\mathbb{I}_x$. Applying (8) and (14) and using definition (1) we get

$$\mathbb{I}_x = \left(I_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma} t^{\rho-1} E_{\theta,\vartheta}^{\lambda,c}(t,p)\, {}_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x)$$
$$= \left(I_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma} t^{\rho-1} \sum_{n=0}^{\infty} \frac{B_p(\lambda+n, c-\lambda)}{B(\lambda, c-\lambda)} \frac{(c)_n}{\Gamma(\theta n+\vartheta)} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-s)^k \left(\frac{t}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\right)(x).$$

By changing the order of integration and summation, we get

$$\mathbb{I}_x = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{B_p(\lambda+n,c-\lambda)}{B(\lambda,c-\lambda)} \frac{(c)_n}{\Gamma(\theta n+\vartheta) n!} \frac{(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)} \left(I_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma} t^{(\rho+2k+q+n+1)-1}\right)(x)$$

Hence replacing ρ by $\rho+2k+q+n+1$ in Lemma 1, we get

$$\mathbb{I}_x = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{B_p(\lambda+n, c-\lambda)}{\Gamma(\lambda)\Gamma(c-\lambda)} \frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)} \frac{(-s)^k \left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$
$$\times \frac{\Gamma(\rho+2k+q+n+1)\Gamma(\rho+2k+q+n+1+\gamma-\eta-\eta'-\sigma)\Gamma(\rho+2k+q+n+1+\sigma'-\eta')}{\Gamma(\rho+2k+q+n+1+\sigma')\Gamma(\rho+2k+q+n+1+\gamma-\eta-\eta')\Gamma(\rho+2k+q+n+1+\gamma-\eta'-\sigma)}$$
$$\times \frac{x^{\rho+2k+q+n-\eta-\eta'+\gamma}}{n!}$$
$$= \frac{x^{\rho+q-\eta-\eta'+\gamma}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k \left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)} \sum_{n=0}^{\infty} \frac{B_p(\lambda+n, c-\lambda)}{\Gamma(c-\lambda)} \frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)}$$
$$\times \frac{\Gamma(\rho+2k+q+n+1)\Gamma(\rho+2k+q+n+1+\gamma-\eta-\eta'-\sigma)\Gamma(\rho+2k+q+n+1+\sigma'-\eta')}{\Gamma(\rho+2k+q+n+1+\sigma')\Gamma(\rho+2k+q+n+1+\gamma-\eta-\eta')\Gamma(\rho+2k+q+n+1+\gamma-\eta'-\sigma)} \frac{x^n}{n!}$$

whose last summation, in view of (9), is easily seen at the expression in (17). This completes the proof of the theorem. □

Corollary 1. *Let $\eta, \sigma, , q, r, s, c, \vartheta, \gamma, \rho \in \mathbb{C}$ with $\Re(\gamma) > 0$ and $\Re(\rho) > max\{0, \Re(\sigma-\gamma)\}$ and $p \geqslant 0, \alpha > 0, \xi > 0; a$ and β is arbitrary. Also, Let $x \in \Re^+$. Then*

$$\left(I_{0,+}^{\eta,\sigma,\gamma} t^{\rho-1} E_{\theta,\vartheta}^{\lambda,c}(t,p)_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x) = \frac{(x)^{\rho+q-\eta+\gamma}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k \left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$
$$\times_4 \Psi_4 \left[\begin{matrix}(c,1),(\rho+2k+q+1+\gamma-\eta-\sigma,1),(\rho+2k+q+1,1),(\lambda,1);\\((\vartheta,\theta),(\rho+2k+q+1+\gamma-\eta,1),(\rho+2k+q+1+\gamma-\sigma,1),(c,1);\end{matrix} x;p\right].$$

Corollary 2. *Let $\alpha = a = 1, \beta = \frac{3}{2}$ and $\xi = 1$. Then above Theorem 1 is reduced to*

$$\left(I_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma} t^{\rho-1}\, E_{\theta,\vartheta}^{\lambda,c}(t,p)\, \mathcal{H}_{q,r,s}(t)\right)(x) = \frac{(x)^{\rho+q-\eta-\eta'+\gamma}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k \left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+q+\frac{r+2}{2}\right)}$$
$$\times_5\Psi_5 \left[\begin{matrix}(c,1),(\rho+2k+q+1,1),(\rho+2k+q+1+\gamma-\eta-\eta'-\sigma,1),(\rho+2k+q+1+\sigma'-\eta',1),(\lambda,1);\\((\vartheta,\theta),(\rho+\sigma'+2k+q+1,1),(\rho+2k+q+1+\gamma-\eta-\eta',1),(\rho+2k+q+1+\gamma-\eta'-\sigma,1),(c,1);\end{matrix} x;p\right].$$

Theorem 2. *Let $\eta, \eta', \sigma, \sigma', q, r, s, c, \vartheta, \gamma, \rho \in \mathbb{C}$ with $\Re(\gamma) > 0$ and $\Re(\rho) > max\{\Re(\rho), \Re(-\eta-\eta'+\gamma), \Re(-\eta-\sigma'+\gamma)\}$ and $p \geqslant 0, \alpha > 0, \xi > 0; a$ and β is arbitrary and let $x \in \Re^+$. Then*

$$\left(I_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma} t^{-\rho} E_{\theta,\vartheta}^{\lambda,c}(t,p)_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x) = \frac{(x)^{-\eta-\eta'+\gamma-\rho+q+1}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k \left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$
$$\times_5\Psi_5 \left[\begin{matrix}(c,1),(\rho-\sigma-2k-q-1,1),(\rho-2k-q-1-\gamma+\eta+\eta',1),(\rho-2k-q-1+\sigma'+\eta-\gamma,1),(\lambda,1);\\((\vartheta,\theta),(\rho-2k-q-1,1),(\rho-2k-q-1+\eta-\sigma,1),(\rho-2k-q-1+\sigma'+\eta+\eta'-\gamma,1),(c,1);\end{matrix} x;p\right]. \quad (18)$$

Proof. Similarly to Theorem 1 let the left-hand side of (18) be denoted by $\mathbb{I}_x$. Applying (8) and (14) and using definition (1) we get

$$\mathbb{I}_x = \left(I_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma} t^{-\rho} E_{\theta,\vartheta}^{\lambda,c}(t,p)_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x)$$

$$= \left(I_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma} t^{-\rho} \sum_{n=0}^{\infty} \frac{B_p(\lambda+n, c-\lambda)}{B(\lambda, c-\lambda)} \frac{(c)_n}{\Gamma(\theta n+\vartheta)} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-s)^k \left(\frac{t}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\right)(x).$$

By changing the order of integration and summation, we get

$$\mathbb{I}_x = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{B_p(\lambda+n, c-\lambda)}{B(\lambda, c-\lambda)} \frac{(c)_n}{\Gamma(\theta n+\vartheta) n!} . \frac{(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)} \left(I_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma} t^{-(\rho-2k-q-n-1)}\right)(x).$$

Hence replacing ρ by $\rho-2k-q-n-1$ in Lemma 2, we get

$$\mathbb{I}_x = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{B_p(\lambda+n, c-\lambda)}{\Gamma(\lambda)\Gamma(c-\lambda)} \frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)} \frac{(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$
$$\times \frac{\Gamma(-\sigma+\rho-2k-q-n-1)\Gamma(\rho-2k-q-n-1-\gamma+\eta+\eta')\Gamma(\eta+\rho-2k-q-n-1+\sigma'-\gamma)}{\Gamma(\rho-2k-q-n-1)\Gamma(\rho-2k-q-n-1-\sigma+\eta)\Gamma(\rho+\eta+\eta'+\sigma'-\gamma-2k-q-n-1)}$$
$$\times \frac{x^{-\rho+2k+q+n+1-\eta-\eta'+\gamma}}{n!}$$
$$= \frac{x^{-\rho+q+1-\eta-\eta'+\gamma}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)} \sum_{n=0}^{\infty} \frac{B_p(\lambda+n, c-\lambda)}{\Gamma(c-\lambda)} \frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)}$$
$$\times \frac{\Gamma(-\sigma+\rho-2k-q-n-1)\Gamma(\rho-2k-q-n-1-\gamma+\eta+\eta')\Gamma(\eta+\rho-2k-q-n-1+\sigma'-\gamma)}{\Gamma(\rho-2k-q-n-1)\Gamma(\rho-2k-q-n-1-\sigma+\eta)\Gamma(\rho+\eta+\eta'+\sigma'-\gamma-2k-q-n-1)} \frac{x^n}{n!}$$

whose last summation, in view of (9), is easily seen at the expression in (18). This completes the proof of the theorem. □

Corollary 3. *Let $\eta, \sigma, , q, r, s, c, \vartheta, \gamma, \rho \in \mathbb{C}$ with $\Re(\gamma) > 0$ and $\Re(\rho) > max\{0, \Re(\sigma-\gamma)\}$ and $p \geqslant 0, \alpha > 0, \xi > 0; a$ and β is arbitrary. Also, let $x \in \Re^+$. Then*

$$\left(I_{0,+}^{\eta,\sigma,\gamma} t^{\rho-1} E_{\theta,\vartheta}^{\lambda,c}(t,p)\, {}_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x) = \frac{(x)^{-\rho+q-\eta+\gamma+1}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$
$$\times {}_4\Psi_4\left[\begin{matrix}(c,1),(-\sigma+\rho-2k-q-1,1),(\eta+\rho-2k-q-1-\gamma,1),(\lambda,1);\\((\vartheta,\theta),(\rho-2k-q-1,1),(\rho-2k-q-1-\sigma+\eta,1),(c,1);\end{matrix} x;p\right].$$

Corollary 4. *Let $\alpha = a = 1, \beta = \frac{3}{2}$ and $\xi = 1$. Then above Theorem 2 is reduced*

$$\left(I_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma} t^{\rho-1}\, E_{\theta,\vartheta}^{\lambda,c}(t,p)\, \mathcal{H}_{q,r,s}(t)\right)(x) = \frac{(x)^{-\rho+q-\eta-\eta'+\gamma+1}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(k+\frac{3}{2})\Gamma\left(k+q+\frac{r+2}{2}\right)}$$
$$\times {}_5\Psi_5\left[\begin{matrix}(c,1),(-\sigma+\rho-2k-q-1,1),(\rho-2k-q-1-\gamma+\eta+\eta',1),(\rho-2k-q-1+\sigma'+\eta-\gamma,1),(\lambda,1);\\((\vartheta,\theta),(\rho-2k-q-1,1),(\rho-2k-q-1+\eta-\sigma,1),(\rho-2k-q-1+\sigma'+\eta+\eta'-\gamma,1),(c,1);\end{matrix} x;p\right].$$

3 Marichev-Saigo-Maeda (MSM) fractional differential representation involving product of Extended Mittag-Leffler Function (EMLF) and generalized Galue Type Struve Function (GTSF)

Here we present product of Extended Mittag-Leffler Function (EMLF) and generalized Galue Type Struve Function (GTSF) in view of the MSM fractional differential representations and

consider some particular cases
Here we recall the following lemma [see (Kilbas & Sebastian, 2008)]

Lemma 3. *Let $\eta,\eta',\sigma,\sigma',\gamma,\rho\in\mathbb{C}$ such that $\Re(\rho)>max\{0,\Re(-\eta+\sigma'),\Re(-\eta-\eta'-\sigma+\gamma\}$. Then*

$$\left(\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\eta-\sigma)\Gamma(\eta+\eta'+\sigma'-\gamma+\rho)}{\Gamma(\rho-\sigma)\Gamma(\rho-\gamma+\eta+\eta')\Gamma(\rho-\gamma+\eta+\sigma')}x^{\rho+\eta+\eta'-\gamma-1} \tag{19}$$

Lemma 4. *Let $\eta,\eta',\sigma,\sigma',\gamma,\rho\in\mathbb{C}$ such that $\Re(\rho)>max\{\Re(-\sigma'),\Re(\eta'+\sigma-\gamma),\Re(\eta+\eta'-\gamma)+[Re(\gamma)]+1\}$. Then*

$$\left(\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma}t^{-\rho}\right)(x)=\frac{\Gamma(\rho+\sigma')\Gamma(-\eta-\eta'+\gamma+\rho)\Gamma(-\eta'-\sigma+\gamma+\rho)}{\Gamma(\rho)\Gamma(-\eta'+\sigma'+\rho)\Gamma(-\eta-\eta'-\sigma+\gamma+\rho)}x^{\eta+\eta'-\gamma-\rho}. \tag{20}$$

Theorem 3. *Let $\eta,\eta',\sigma,\sigma',q,r,s,c,\vartheta,\gamma,\rho\in\mathbb{C}$ with $\Re(\gamma)>0$ and $\Re(\rho)>max\{0,\Re(-\eta+\sigma),\Re(-\eta-\eta'-\sigma'+\gamma)\}$ and $\alpha>0,\xi>0;a$ and β is arbitrary. Then*

$$\left(\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma}t^{\rho-1}E_{\theta,\vartheta}^{\lambda,c}(t,p)\,{}_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x)=\frac{(x)^{\rho+q+\eta+\eta'-\gamma}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$

$$\times{}_5\Psi_5\left[\begin{matrix}(c,1),(\rho+2k+q+1,1),(\rho+2k+q+1+\gamma+\eta-\sigma,1),(\rho+2k+q+1+\sigma'+\eta+\eta'-\gamma,1),(\lambda,1);\\((\vartheta,\theta),(\rho+2k+q+1-\sigma,1),(\rho+2k+q+1-\gamma+\eta+\eta',1),(\rho+2k+q+1-\gamma+\eta+\sigma',1),(c,1);\end{matrix}x;p\right] \tag{21}$$

Proof. Let the left-hand side of (21) be denoted by $\mathbb{D}_x$. Applying (8) and (14) and using definition (1) we get

$$\mathbb{D}_x=\left(\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma}t^{\rho-1}E_{\theta,\vartheta}^{\lambda,c}(t,p)\,{}_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x)$$
$$=\left(\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma}t^{\rho-1}\sum_{n=0}^{\infty}\frac{B_p(\lambda+n,c-\lambda)}{B(\lambda,c-\lambda)}\frac{(c)_n}{\Gamma(\theta n+\vartheta)}\frac{t^n}{n!}\sum_{k=0}^{\infty}\frac{(-s)^k\left(\frac{t}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\right)(x)$$

By changing the differential and summation order, we get

$$\mathbb{D}_x=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_p(\lambda+n,c-\lambda)}{B(\lambda,c-\lambda)}\frac{(c)_n}{\Gamma(\theta n+\vartheta)n!}\frac{(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\left(\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma}t^{(\rho+2k+q+n+1)-1}\right)(x)$$

Hence replacing ρ by $\rho+2k+q+n+1$ in lemma 3, we get

$$\mathbb{D}_x=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_p(\lambda+n,c-\lambda)}{\Gamma(\lambda)\Gamma(c-\lambda)}\frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)}\frac{(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$
$$\times\frac{\Gamma(\rho+2k+q+n+1)\Gamma(\rho+2k+q+n+1+\eta-\sigma)\Gamma(\rho+2k+q+n+1+\eta+\eta'+\sigma'-\gamma)}{\Gamma(\rho+2k+q+n+1-\sigma)\Gamma(\rho+2k+q+n+1-\gamma+\eta+\eta')\Gamma(\rho+2k+q+n+1+\eta+\sigma'-\gamma)}$$
$$\times\frac{x^{\rho+2k+q+n+\eta+\eta'-\gamma}}{n!}$$
$$=\frac{x^{\rho+q+\eta+\eta'-\gamma}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\sum_{n=0}^{\infty}\frac{B_p(\lambda+n,c-\lambda)}{\Gamma(c-\lambda)}\frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)}$$
$$\times\frac{\Gamma(\rho+2k+q+n+1)\Gamma(\rho+2k+q+n+1+\eta-\sigma)\Gamma(\rho+2k+q+n+1+\sigma'+\eta+\eta'-\gamma)}{\Gamma(\rho+2k+q+n+1-\sigma)\Gamma(\rho+2k+q+n+1-\gamma+\eta+\eta')\Gamma(\rho+2k+q+n+1-\gamma+\eta+\sigma')}\frac{x^n}{n!}$$

whose last summation, in view of (9), is easily seen at the expression in (21). This completes the proof of the theorem. □

Corollary 5. *Let $\eta,\sigma,q,r,s,c,\vartheta,\gamma,\rho\in\mathbb{C}$ with $\Re(\gamma)>0$ and $\Re(\rho)>max\{0,\Re(-\eta),\Re(-\eta-\sigma+\gamma)\}$ and $\alpha>0,\xi>0;a$ and β is arbitrary. Then*

$$\left(\mathcal{D}_{0,+}^{\eta,\sigma,\gamma}t^{\rho-1}E_{\theta,\vartheta}^{\lambda,c}(t,p)\,{}_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x)=\frac{(x)^{\rho+q+\eta-\gamma}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$
$$\times{}_4\Psi_4\left[\begin{matrix}(c,1),(\rho+2k+q+1,1),(\rho+2k+q+1+\eta-\sigma,1),(\lambda,1);\\((\vartheta,\theta),(\rho+2k+q+1-\sigma,1),(\rho+2k+q+1+\eta-\gamma,1),(c,1);\end{matrix}x;p\right].$$

Corollary 6. *Let $\alpha=a=1,\beta=\frac{3}{2}$ and $\xi=1$. Then Theorem 3 is reduced to*

$$\left(\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma}t^{\rho-1}\,E_{\theta,\vartheta}^{\lambda,c}(t,p)\,\mathcal{H}_{q,r,s}(t)\right)(x)=\frac{(x)^{\rho+q+\eta+\eta'-\gamma}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+q+\frac{r+2}{2}\right)}$$
$$\times{}_5\Psi_5\left[\begin{matrix}(c,1),(\rho+2k+q+1,1),(\rho+2k+q+1+\gamma+\eta-\sigma,1),(\rho+2k+q+1+\sigma'+\eta+\eta'-\gamma,1),(\lambda,1);\\((\vartheta,\theta),(\rho+2k+q+1-\sigma,1),(\rho+2k+q+1-\gamma+\eta+\eta',1),(\rho+2k+q+1-\gamma+\eta+\sigma',1),(c,1);\end{matrix}x;p\right]$$

Theorem 4. *Let $\eta,\eta',\sigma,\sigma',q,r,s,c,\vartheta,\gamma,\rho\in\mathbb{C}$ with $\Re(\gamma)>0$ and $\Re(\rho)>max\{0,\Re(-\sigma'),\Re(\eta'+\sigma-\gamma),\Re(\eta+\eta'-\gamma)+[\Re(\gamma+1]\}$ and $\alpha>0,\xi>0;a$ and β is arbitrary, then*

$$\left(\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma}t^{-\rho}E_{\theta,\vartheta}^{\lambda,c}(t,p)\,{}_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x)=\frac{(x)^{-\rho+q+\eta+\eta'-\gamma+1}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}.$$
$$\times{}_5\Psi_5\left[\begin{matrix}(c,1),(\rho+\sigma'-2k-q-1,1),(\rho-2k-q-1-\eta-\eta'+\gamma,1),(\rho-2k-q-1-\eta'-\sigma+\gamma,1),(\lambda,1);\\((\vartheta,\theta),(\rho-2k-q-1,1),(\rho-2k-q-1-\eta'+\sigma',1),(\rho-2k-q-1+\gamma-\eta-\eta'-\sigma,1),(c,1);\end{matrix}x;p\right]\quad(22)$$

Proof. Let the left-hand side of (22) be denoted by $\mathbb{D}_x$. Applying (8) and (14) and using definition (1) we get

$$\mathbb{D}_x=\left(\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma}t^{-\rho}E_{\theta,\vartheta}^{\lambda,c}(t,p)\,{}_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x)$$
$$=\left(\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma}t^{-\rho}\sum_{n=0}^{\infty}\frac{B_p(\lambda+n,c-\lambda)}{B(\lambda,c-\lambda)}\frac{(c)_n}{\Gamma(\theta n+\vartheta)}\frac{t^n}{n!}\sum_{k=0}^{\infty}\frac{(-s)^k\left(\frac{t}{2}\right)^{2k+q+1}}{\Gamma(\alpha\,k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\right)(x).$$

By changing the differential and summation order, we get

$$\mathbb{D}_x=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_p(\lambda+n,c-\lambda)}{B(\lambda,c-\lambda)}\frac{(c)_n}{\Gamma(\theta n+\vartheta)n!}\frac{(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\left(\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma}t^{-(\rho-2k-q-n-1)}\right)(x).$$

Hence replacing ρ by $\rho-2k-q-n-1$ in lemma 4, we get

$$\mathbb{D}_x=\sum_{n=0}^{\infty}\sum_{k=0}\frac{B_p(\lambda+n,c-\lambda)}{\Gamma(\lambda)\Gamma(c-\lambda)}\frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)}\frac{(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$
$$\times\frac{\Gamma(\rho-2k-q-n-1+\sigma')\Gamma(\rho-2k-q-n-1-\eta-\eta'+\gamma)\Gamma(\rho-2k-q-n-1-\eta'-\sigma+\gamma)}{\Gamma(\rho-2k-q-n-1)\Gamma(\rho-2k-q-n-1-\eta'+\sigma')\Gamma(\rho-2k-q-n-1-\eta-\eta'-\sigma+\gamma)}$$
$$\times\frac{x^{-\rho+2k+q+n+1+\eta+\eta'-\gamma}}{n!}$$
$$=\frac{x^{-\rho+q+\eta+\eta'-\gamma+1}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(x^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\sum_{n=0}^{\infty}\frac{B_p(\lambda+n,c-\lambda)}{\Gamma(\lambda)\Gamma(c-\lambda)}\frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)}$$
$$\times\frac{\Gamma(\rho-2k-q-n-1+\sigma')\Gamma(\rho-2k-q-n-1-\eta-\eta'+\gamma)\Gamma(\rho-2k-q-n-1-\eta'-\sigma+\gamma)}{\Gamma(\rho-2k-q-n-1)\Gamma(\rho-2k-q-n-1-\eta'+\sigma')\Gamma(\rho-2k-q-n-1-\eta-\eta'-\sigma+\gamma)}\frac{x^n}{n!}$$

whose last summation, in view of (9), is easily seen at the expression in (22). This completes the proof of the theorem. □

Corollary 7. *Let* $\eta, \sigma, , q, r, s, c, \vartheta, \gamma, \rho \in \mathbb{C}$ *with* $\Re(\gamma) > 0$ *and* $\Re(\rho) > max\{0, , \Re(\sigma - \gamma), \Re(\eta - \gamma) + [\Re(\gamma + 1]\}$ *and* $\alpha > 0, \xi > 0; a$ *and* β *is arbitrary. Then*

$$\left(\mathcal{D}_{0,-}^{\eta,\sigma,\gamma} t^{-\rho} E_{\theta,\vartheta}^{\lambda,c}(t,p)\ {}_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x) = \frac{(x)^{-\rho+q+\eta-\gamma+1}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k \left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$

$$\times {}_4\Psi_4 \left[\begin{matrix}(c,1), (-\eta+\gamma+\rho-2k-q-1,1), (\rho-2k-q-1-\sigma+\gamma,1), (\lambda,1); \\ ((\vartheta,\theta), (\rho-2k-q-1,1), (\rho-2k-q-1-\eta-\sigma+\gamma,1), (c,1);\end{matrix} x;p\right].$$

Corollary 8. *Let* $\alpha = a = 1, \beta = \frac{3}{2}$ *and* $\xi = 1$. *Then Theorem 4 turns to*

$$\left(\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma} t^{-\rho}\ E_{\theta,\vartheta}^{\lambda,c}(t,p)\ \mathcal{H}_{q,r,s}(t)\right)(x) = \frac{(x)^{-\rho+q+\eta+\eta'-\gamma+1}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k \left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+q+\frac{r+2}{2}\right)}$$

$$\times {}_5\Psi_5 \left[\begin{matrix}(c,1), (\rho+\sigma'-2k-q-1,1), (\rho-2k-q-1-\eta-\eta'+\gamma,1), (\rho-2k-q-1-\eta'-\sigma+\gamma,1), (\lambda,1); \\ ((\vartheta,\theta), (\rho-2k-q-1,1), (\rho-2k-q-1-\eta'+\sigma',1), (\rho-2k-q-1+\gamma-\eta-\eta'-\sigma,1), (c,1);\end{matrix} x;p\right].$$

4 Caputo-Type Marichev-Saigo-Maeda (MSM) fractional differential representation involving product of EMLF and GTSF

Here we present product of Extended Mittag-Leffler Function (EMLF) and generalized Galue Type Struve Function (GTSF) in view of Caputo-Type MSM fractional differential representations and consider some particular cases

We need the following lemma [see (Kataria et al., 2015; Araci et al., 2019)]

Lemma 5. *Let* $\eta, \eta', \sigma, \sigma', \gamma, \rho \in \mathbb{C}$ *and* $m = [\Re(\gamma)] + 1$ *with* $\Re(\rho) - m > max\{0, \Re(-\eta + \sigma), \Re(-\eta - \eta' - \sigma' + \gamma)\}$ *and* $p \geqslant 0$. *Then*

$$\left({}^C\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\eta-\sigma-m)\Gamma(\eta+\eta'+\sigma-\gamma+\rho-m)}{\Gamma(\rho-\sigma-m)\Gamma(\rho-\gamma+\eta+\eta')\Gamma(\rho-\gamma+\eta+\sigma'-m)} x^{\rho+\eta+\eta'-\gamma-1}. \tag{23}$$

Lemma 6. *Let* $\eta, \eta', \sigma, \sigma', \gamma, \rho \in \mathbb{C}$ *and* $m = [\Re(\gamma)] + 1$ *with* $\Re(\rho) + m > max\{\Re(-\sigma'), \Re(\eta' + \sigma - \gamma), \Re(\eta + \eta' - \gamma) + [\Re(\gamma)] + 1\}$. *Then*

$$\left({}^C\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma} t^{-\rho}\right)(x) = \frac{\Gamma(\rho+\sigma'+m)\Gamma(-\eta-\eta'+\gamma+\rho)\Gamma(-\eta'-\sigma+\gamma+\rho+m)}{\Gamma(\rho)\Gamma(-\eta'+\sigma'+\rho+m)\Gamma(-\eta-\eta'-\sigma+\gamma+\rho+m)} x^{\eta+\eta'-\gamma-\rho}. \tag{24}$$

Theorem 5. *Let* $\eta, \eta', \sigma, \sigma', \gamma, \rho \in \mathbb{C}$ *and* $m = [\Re(\gamma)+1], \Re(\rho)-m > max\{0, \Re(-\eta+\sigma'), \Re(-\eta-\eta'-\sigma'+\gamma\}$ *and* $p \geqslant 0; \alpha > 0, \xi > 0; a \in N$ *and* β *is an arbitrary. Also, let* $x \in \Re^+$. *Then*

$$\left({}^C\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma} t^{\rho-1} E_{\theta,\vartheta}^{\lambda,c}(t,p)\ {}_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x) = \frac{(x)^{\rho+q+\eta+\eta'-\gamma}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k \left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}$$

$$\times {}_5\Psi_5 \left[\begin{matrix}(c,1), (\rho+2k+q+1,1), (\rho+2k+q+1+\eta-\sigma-m,1), (\rho+2k+q+1+\sigma'+\eta+\eta'-\gamma-m,1), (\lambda,1); \\ ((\vartheta,\theta), (p+2k+q+1-\sigma-m,1), (\rho+2k+q+1-\gamma+\eta+\eta',1), (\rho+2k+q+1-\gamma+\eta+\sigma'-m,1), (c,1);\end{matrix} x;p\right]. \tag{25}$$

Proof. Let the left-hand side of (25) be denoted by ${}^C\mathbb{D}_x$. Applying (8) and (14) and using the definition (1) we get

$$\begin{aligned}{}^C\mathbb{D}_x &= \left(\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma} t^{\rho-1} E_{\theta,\vartheta}^{\lambda,c}(t,p)\ {}_a\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x) \\ &= \left(\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma} t^{\rho-1} \sum_{n=0}^{\infty} \frac{B_p(\lambda+n, c-\lambda)}{B(\lambda, c-\lambda)} \frac{(c)_n}{\Gamma(\theta n+\vartheta)} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-s)^k \left(\frac{t}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\right)(x).\end{aligned}$$

By changing the differential and summation order, we get

$$^{C}\mathbb{D}_x=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_p(\lambda+n,c-\lambda)}{B(\lambda,c-\lambda)}\frac{(c)_n}{\Gamma(\theta n+\vartheta)n!}\frac{(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\left({}^{C}\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma}t^{(\rho+2k+q+n+1)-1}\right)(x).$$

Hence replacing ρ by $\rho+2k+q+n+1$ in lemma 5, we get

$$\begin{aligned}^{C}\mathbb{D}_x=&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_p(\lambda+n,c-\lambda)}{\Gamma(\lambda)\Gamma(c-\lambda)}\frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)}\frac{(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\\&\times\frac{\Gamma(\rho+2k+q+n+1)\Gamma(\rho+2k+q+n+1+\eta-\sigma-m)\Gamma(\rho+2k+q+n+1+\sigma'+\eta+\eta'-\gamma-m)}{\Gamma(\rho+2k+q+n+1-\sigma-m)\Gamma(\rho+2k+q+n+1-\gamma+\eta+\eta')\Gamma(\rho+2k+q+n+1+\eta+\sigma'-\gamma-m)}\\&\times\frac{x^{\rho+2k+q+n+\eta+\eta'-\gamma}}{n!}\\=&\frac{x^{\rho+q+\eta+\eta'-\gamma}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\sum_{n=0}^{\infty}\frac{B_p(\lambda+n,c-\lambda)}{\Gamma(c-\lambda)}\frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)}\\&\times\frac{\Gamma(\rho+2k+q+n+1)\Gamma(\rho+2k+q+n+1+\eta-\sigma-m)\Gamma(\rho+2k+q+n+1+\sigma'+\eta+\eta'-\gamma-m)}{\Gamma(\rho+2k+q+n+1-\sigma-m)\Gamma(\rho+2k+q+n+1-\gamma+\eta+\eta')\Gamma(\rho+2k+q+n+1+\eta+\sigma'-\gamma-m)}\frac{x^n}{n!}\end{aligned}$$

whose last summation, in view of (9), is easily seen at the expression in (25). This completes the proof of the theorem. □

Corollary 9. *Let* $\eta,\sigma,q,r,s,c,\vartheta,\gamma,\rho\in\mathbb{C}$ *and* $m=[\Re(\gamma)+1]$,$\Re(\rho)-m>max\{0,\Re(-\eta),\Re(-\eta+\gamma)\}$ *and* $p\geqslant 0$; $\alpha>0,\xi>0;a$ *and* β *is arbitrary. Also, let* $x\in^{+}$. *Then*

$$\begin{aligned}\left({}^{C}\mathcal{D}_{0,+}^{\eta,\sigma,\gamma}t^{\rho-1}E_{\theta,\vartheta}^{\lambda,c}(t,p)\;{}_{a}\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x)=&\frac{(x)^{\rho+q+\eta-\gamma}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\\&\times{}_4\Psi_4\left[\begin{matrix}(c,1),(\rho+2k+q+1,1),(\rho+2k+q+1+\eta-\sigma-m,1),(\lambda,1);\\((\vartheta,\theta),(\rho+2k+q+1-\sigma-m,1),(\rho+2k+q+1+\eta-\gamma,1),(c,1);\end{matrix}x;p\right]\end{aligned}$$

Corollary 10. *Let* $\alpha=a=1,\beta=\frac{3}{2}$ *and* $\xi=1$. *Then Theorem 5 is reduced*

$$\begin{aligned}\left({}^{C}\mathcal{D}_{0,+}^{\eta,\eta',\sigma,\sigma',\gamma}t^{\rho-1}\;E_{\theta,\vartheta}^{\lambda,c}(t,p)\;\mathcal{H}_{q,r,s}(t)\right)(x)=&\frac{(x)^{\rho+q+\eta+\eta'-\gamma}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(k+\frac{3}{2})\Gamma\left(k+q+\frac{r+2}{2}\right)}\\\times{}_5\Psi_5\left[\begin{matrix}(c,1),(\rho+2k+q+1,1),(\rho+2k+q+1+\eta-\sigma-m,1),(\rho+2k+q+1+\sigma'+\eta+\eta'-\gamma-m,1),(\lambda,1);\\((\vartheta,\theta),(p+2k+q+1-\sigma-m,1),(\rho+2k+q+1-\gamma+\eta+\eta',1),(\rho+2k+q+1-\gamma+\eta+\sigma'-m,1),(c,1);\end{matrix}x;p\right].\end{aligned}$$

Theorem 6. *Let* $\eta,\eta',\sigma,\sigma',q,r,s,c,\vartheta,\gamma,\rho\in\mathbb{C}$ *and* $m=[\Re(\gamma)+1]$ *with* $\Re(\rho)+m>max\{\Re(-\sigma'),\Re(\eta'+\eta'-\gamma)\}$ *and* $p\geqslant 0$,$\alpha>0,\xi>0;a$ *and* β *is arbitrary. Then*

$$\begin{aligned}\left({}^{C}\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma}t^{-\rho}E_{\theta,\vartheta}^{\lambda,c}(t,p)\;{}_{a}\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x)=&\frac{(x)^{-\rho+q+\eta+\eta'-\gamma+1}}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\\\times{}_5\Psi_5\left[\begin{matrix}(c,1),(\sigma'+\rho-2k-q-1+m,1),(\rho-2k-q-1-\eta-\eta'+\gamma,1),(-\eta'-\sigma+\gamma+\rho-2k-q-1+m,1),(\lambda,1);\\((\vartheta,\theta),(\rho-2k-q-1,1),(\rho-2k-q-1-\eta'+\sigma'+m,1),(\rho-2k-q-1+\gamma-\eta-\eta'-\sigma+m,1),(c,1);\end{matrix}x;p\right].\end{aligned}\quad(26)$$

Proof. Let the left-hand side of (26) be denoted by ${}^{C}\mathbb{D}_x$. Applying (8) and (14) and using the definition (1) we get

$$
\begin{aligned}
{}^{C}\mathbb{D}_x &= \left(\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma} t^{-\rho} E_{\theta,\vartheta}^{\lambda,c}(t,p)\, {}_{a}\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x) \\
&= \left(\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma} t^{-\rho} \sum_{n=0}^{\infty} \frac{B_p(\lambda+n, c-\lambda)}{B(\lambda, c-\lambda)} \frac{(c)_n}{\Gamma(\theta n+\vartheta)} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-s)^k \left(\frac{t}{2}\right)^{2k+q+1}}{\Gamma(\alpha\, k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)}\right)(x).
\end{aligned}
$$

By changing the differential and summation order, we get

$$
\mathbb{D}_x = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{B_p(\lambda+n,c-\lambda)}{B(\lambda,c-\lambda)} \frac{(c)_n}{\Gamma(\theta n+\vartheta) n!} \frac{(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)} \left(\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma} t^{-(\rho-2k-q-n-1)}\right)(x).
$$

Hence replacing ρ by $\rho-2k-q-n-1$ in lemma 6, we get

$$
\begin{aligned}
\mathbb{D}_x =& \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{B_p(\lambda+n,c-\lambda)}{\Gamma(\lambda)\Gamma(c-\lambda)} \frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)} \frac{(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)} \\
&\times \frac{\Gamma(\rho-2k-q-n-1+\sigma'+m)\Gamma(\rho-2k-q-n-1-\eta-\eta'+\gamma)\Gamma(\rho-2k-q-n-1-\eta'-\sigma+\gamma+m)}{\Gamma(\rho-2k-q-n-1)\Gamma(\rho-2k-q-n-1-\eta'+\sigma'+m)\Gamma(\rho-2k-q-n-1-\eta-\eta'-\sigma+\gamma+m)} \\
&\times \frac{x^{-\rho+2k+q+n+1+\eta+\eta'-\gamma}}{n!} \\
=& \frac{x^{-\rho+q+\eta+\eta'-\gamma+1}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)} \sum_{n=0}^{\infty} \frac{B_p(\lambda+n,c-\lambda)}{\Gamma(c-\lambda)} \frac{\Gamma(c+n)}{\Gamma(\theta n+\vartheta)} \\
&\times \frac{\Gamma(\rho-2k-q-n-1+\sigma'+m)\Gamma(\rho-2k-q-n-1-\eta-\eta'+\gamma)\Gamma(-2k-q-n-1-\eta'-\sigma+\gamma+m)}{\Gamma(\rho-2k-q-n-1)\Gamma(\rho-2k-q-n-1-\eta'+\sigma'+m)\Gamma(\rho-2k-q-n-1-\eta-\eta'-\sigma+\gamma+m)} \frac{x^n}{n!}.
\end{aligned}
$$

whose last summation, in view of (9), is easily seen at the expression in (26). This completes the proof of the theorem. □

Corollary 11. *Let $\eta,\sigma,,q,r,s,c,\vartheta,\gamma,\rho \in \mathbb{C}$ and $m = [\Re(\gamma)+1]$ with $\Re(\gamma)+m > 0$ and $\Re(\rho) > max\{0, \Re(\eta-\gamma)+m\}$ and $p \geqslant 0; \alpha > 0, \xi > 0; a$ and β is arbitrary. Also, let $x \in \Re^+$.Then*

$$
\begin{aligned}
&\left({}^{C}\mathcal{D}_{0,-}^{\eta,\sigma,\gamma} t^{-\rho} E_{\theta,\vartheta}^{\lambda,c}(t,p)\, {}_{a}\mathcal{W}_{q,r,s,\xi}^{\alpha,\beta}(t)\right)(x) = \frac{(x)^{-\rho+q+\eta-\gamma+1}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(\alpha k+\beta)\Gamma\left(ak+\frac{q}{\xi}+\frac{r+2}{2}\right)} \\
&\quad\times {}_4\Psi_4\left[\begin{matrix}(c,1),(-\eta+\gamma+\rho-2k-q-1,1),(\rho-2k-q-1-\sigma+\gamma+m,1),(\lambda,1); \\ ((\vartheta,\theta),(\rho-2k-q-1,1),(\rho-2k-q-1-\eta-\sigma+\gamma+m,1),(c,1);\end{matrix}\ x;p\right]
\end{aligned}
$$

Corollary 12. *Let $\alpha = a = 1, \beta = \frac{3}{2}$ and $\xi = 1$. Then Theorem 6 is reduced*

$$
\begin{aligned}
&\left({}^{C}\mathcal{D}_{0,-}^{\eta,\eta',\sigma,\sigma',\gamma} t^{-\rho}\, E_{\theta,\vartheta}^{\lambda,c}(t,p)\, \mathcal{H}_{q,r,s}(t)\right)(x) = \frac{(x)^{-\rho+q+\eta+\eta'-\gamma+1}}{\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(x)^{2k}(-s)^k\left(\frac{1}{2}\right)^{2k+q+1}}{\Gamma(k+\frac{3}{2})\Gamma\left(k+q+\frac{r+2}{2}\right)} \\
&\times {}_5\Psi_5\left[\begin{matrix}(c,1),(\sigma'+\rho-2k-q-1+m,1),(\rho-2k-q-1-\eta-\eta'+\gamma,1),(-\eta'-\sigma+\gamma\rho-2k-q-1+m,1),(\lambda,1); \\ ((\vartheta,\theta),(\rho-2k-q-1,1),(\rho-2k-q-1-\eta'+\sigma'+m,1),(\rho-2k-q-1+\gamma-\eta-\eta'-\sigma+m,1),(c,1);\end{matrix}\ x;p\right].
\end{aligned}
$$

5 Conclusion

In the paper we established generalized fractional formulas to derive numerous results. The fractional integral and differential formulas (of Marichev–Saigo–Maeda type) involving the product of extended Mittag-Leffler function and generalized Galue Type Struve Function developed in this paper will be very useful and are general in character and likely to find some applications.

References

Araci, S., Rahman, G., Gaffar, A., Azeema, Nisar, K.S. (2019). Fractional calculus of extended Mittag-Leffler function and its applications to statistical distribution. *Mathematics, 7*, 248.

Baricz, A. (2010). *Generalized Bessel functions of the first kind, Lecture Notes in Mathematics.* Springer, Berlin.

Bhowmick, K.N. (1963). A generalized Struve's function and its recurrence formula. *Vijnana Parishad Anusandhan Patrika*, 6, 1-11.

Bhowmick, K. N. (1962). Some relations between a generalized Struve's function and hypergeometric functions. *Vijnana Parishad Anusandhan Patrika*, 5, 93-99.

Chaudhry, M.A, Qadir, A.M., Rafique, S.M. (1997). Extension of Euler's beta function. *J. Comput. Appl. Math.*, 78, 19-32.

Dorrego, G.A., Cerutti, R.A. (2012). The k-Mittag-Leffler function. *Int. J. Contemp. Math. Sci.*, 7, 705-716.

Galue, L. (2003). A generalized Bessel function, *Int. Transforms Spec. Funct.*. 14, 395-401.

Gorenflo, R., Mainardi, F., Srivastava, H.M. (1998). Special functions in fractional relaxation oscillation and fractional diffusion-wave phenomena. *In Proceedings of the Eighth International Colloquium on Differential Equations*; VSP Publishers: London, UK, 195-202.

Gorenflo, R., Kilbas, A.A., Rogosin, S.V. (1998). On the generalized Mittag-Lefler type functions. *Integral Transforms Spec. Funct.*, 7, 215-224.

Hilfer, R. (2000). *Applications of Fractional Calculus in Physics.* World Scientific, Singapore, 87-130.

Kanth, B.N.(1981). Integrals involving generalized Struve's function. *Nepali Math. Sci. Rep.*, 6, 61-64.

Kataria, K. K., Vellaisamy, P. (2015). The generalized k-Wright function and Marichev-Saigo-Maeda fractional operators. *J. Anal.*, 23, 75-87

Kilbas, A.A., Sebastian, N. (2008). Generalized fractional integration of Bessel function of the first kind. *Integral Transform. Spec. Funct.*, 19, 869-883

Kilbas, A.A., Srivastava, H.M., Trujillo, J.J. (2006) *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematical Studies, 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York.

Kiryakova, V. (1997). All the special functions are fractional differintegrals of elementary functions. *Journal of Physics A: Mathematical and General*, *30*(14), 5085-5103.

Kiryakova, V. (1993). *Generalized Fractional Calculus and Applications.* Longman Scientific and Technical, Harlow (Essex).

Kiryakova, V. (2006). On two Saigo's fractional integral operators in the class of univalent functions. *Fract. Calc. Appl. Anal.*, *9*, 160-176.

Marichev, O.I. (1974). Volterra equation of Mellin convolution type with a Horn function in the kernel. *Izvestiya Akademii Nauk BSSR, Seriya Fiziko-Matematicheskikh Nauk*, 1, 128-129.

Miller, K.S., Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations.* John Wiley & Sons, New York, NY, USA.

Mittag-Leffler, G.M. (1903). Sur la nouvelle function $E_x(x)$. *C.R. Acad. Sci. Paris*, 137, 554-558.

Mondal, S.R., Nisar, K.S. (2014). Marichev - Saigo - Maeda fractional integration operators involving generalized Bessel function. *Math. Probl. Engg.*, Article ID 270093.

Mondal, S.R., Swaminathan, A. (2012). Geometric properties of generalized Bessel function. *Bull. Malays. Math. Sci. Soc.*, *35*(1), 179-194.

Nisar, K.S., Atangana, A. (2016). Exact solution of fractional kinetic equation in terms of Struve function. *arXiv:1611.09154 [math.CA].*

Nisar, K.S., Boleanu, D., Qureshi, M. M.(2016). Fractional calculus and applications of generalized Struve function. *Doi 10.1186/540064-016-2560-3*, 1-13.

Nisar, K.S., Purohit, S.D., Mondal, S.R. (2016). Generalized fractional kinetic equations involving generalized Struve function of first kind. *J. King Saud Univ. Sci.*, *28*(2), 161-167.

Oldham, K.B., Spanier , J. (1974). *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*. Academic Press, New York and London.

Orhan, H., Yagmur, N. (2014). Geometric properties of generalized Struve functions. *Ann. Alexandria Loan Cuza Univ-Math*, doi: 10.2478/aicu-2014-007.

Orhan, H., Yagmur, N.(2013). Star likeness and convexity of generalized Struve function. *Abstr. Appl. Anal.*, 6, Art. ID-954513.

Ozarslan, M.A., Yilmaz, B. (2014). The extended Mittag-Leffler function and its properties. *J. Inequal. Appl.*, 85.

Prabhakar, T.R. (1971). A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math. J.*, 19, 7-15.

Rahman, G., Agarwal, P., Mubeen, S., Arshad, M. (2018). Fractional integral operators involving extended Mittag-Leffler function as its Kernel. *Boletin de la Sociedad Matemática Mexicana*, *24*(2), 381-392.

Rahman, G., Baleanu, D., Al-Qurashi, M., Purohit, S.D., Mubeen, S., Arshad, M. (2017). The extended Mittag-Leffler function via fractional calculus. *J. Nonlinear Sci. Appl.*, 10, 4244-4253.

Saigo, M. (1979). A certain boundary value problem for the Euler-Darboux equation I. *Math. Japonica*, 24, 377-385.

Saigo, M. (1980). A certain boundary value problem for the Euler-Darboux equation II. *Math. Japonica*, 25, 211-220.

Saigo, M. (1978). A remark on integral operators involving the Gauss hypergeometric functions. *Math. Rep. Kyushu Univ.*, 11, 135-143.

Saigo, M., Maeda, N. (1998). More generalization of fractional calculus. In T*ransform Methods & Special Functions*. Bulgarian Academy of Sciences: Sofia, Bulgaria, 96, 386-400.

Samko, S.G., Kilbas, A.A., Marchev, O.I. (1993). *Fractional integrals and derivatives: Theory and applications*. Gordon Breach.

Sharma, S.C., Devi, M. (2015). Certain properties of extended Wright generalized hypergeometric function. *Ann. Pure Appl. Math.*, 9, 45-51.

Singh, R.P. (1985). Generalized Struve's function and its recurrence equations. *Vijnana Parishad Anusandhan Patrika*, 28, 62-66.

Singh, R.P. (1974). Generalized Struve's function and its recurrence relations. *Ranchi Univ. Math J.*, 5, 67-75.

Singh, R.P. (1989). Infinite integrals involving generalized Struve's function. *Math Ed. (Siwan)*, 23, 30-36.

Singh, R.P. (1988) Some integrals representation of generalized Struve's function, *Math Ed. (Siwan)*, 22, 91-94.

Srivastava, H.M., Choi, J. (2012). *Zeta and q-Zeta Functions and Associated Series and Integrals.* Elsevier Science Publishers, Amsterdam, London and New York.

Srivastava, H.M., Lin, S.D. Wang, P.Y. (2006). Some fractional calculus results for the H-function associated with a class of Feynman integrals. *Russian Journal of Mathematical Physics*, *13*(1), 94-100.

Srivastava, M., Karlson, P.W. (1985). *Multiple gaussion hypergeometric series.* Ellis Horwood Limited, New York.

Wiman, A. (1905). Über den Fundamentalsatz in der Theorie der Funktionen $E_\alpha(x)$. *Acta Math.* 29, 191-201.